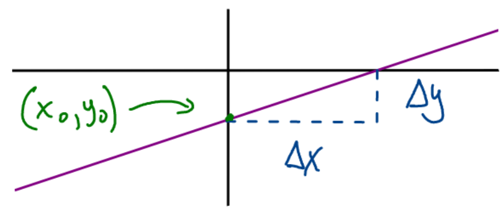
LINES AND PLANES

Recall: In $\mathbb{R}^2$, lines are determined by two pieces of information: ① A point ② A slope.

We think of slope as giving the direction of the line. We know the slope as $\frac{\Delta y}{\Delta x}$.



This means we can rewrite a line formally as $\ell = \{ (x_0 + t\Delta x, y_0 + t\Delta y) : t \in \mathbb{R} \}$.

In $\mathbb{R}^3$, this is how we think of lines!

A line is determined by ① a point and ② a direction vector.

Every line can be expressed by ($\vec{p} = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle a, b, c \rangle$):

① $\ell = \{ \vec{p} + t\vec{v} : t \in \mathbb{R} \}$ (Set notation)

② $\vec{r}(t) = \vec{p} + t\vec{v}$ (vector notation)

③ $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$ (parametric equations)

④ $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ (Symmetric equations)

⇝ NB: If $a = 0$ (for example), we use $x = x_0$ ...

A **line segment** between two points $A, B \in \mathbb{R}^3$ is the set $\overrightarrow{AB} = \{(1-t)\,\overrightarrow{OA} + t\,\overrightarrow{OB} : t \in [0,1]\}$.

Ex: Write the line $\ell: \frac{x-3}{5} = \frac{y+1}{2} = \frac{z-6}{6}$ in the other forms.

Sol: We note $x_0 = 3$, $y_0 = -1$, $z_0 = 6$, $a = 5$, $b = 2$, $c = 6$.

Hence $\vec{p} = \langle 3, -1, 6 \rangle$, $\vec{v} = \langle 5, 2, 6 \rangle$.

Set notation: $\ell = \{ \langle 3,-1,6 \rangle + t\langle 5,2,6 \rangle : t \in \mathbb{R} \}$

Vector Equation: $\vec{r}(t) = \langle 3,-1,6 \rangle + t\langle 5,2,6 \rangle = \langle 3+5t, -1+2t, 6+6t \rangle$

Parametric Equations:
$$\begin{cases} x = 3 + 5t \\ y = -1 + 2t \\ z = 6 + 6t \end{cases}$$

$\blacksquare$

Ex: Do the given lines intersect?

ⓐ $\vec{r}_1(t) = \langle 1,-2,4\rangle + t\langle 1,3,-1\rangle$, $\vec{r}_2(t) = \langle 0,3,-3\rangle + t\langle 2,1,4\rangle$.

Sol: $\vec{r}_1(t) = \langle 1+t, -2+3t, 4-t\rangle$, $\vec{r}_2(t) = \langle 2t, 3+t, -3+4t\rangle$

To see if $\vec{r}_1(t)$ intersects $\vec{r}_2(t)$, we'll solve (attempt!)

$$\boxed{\vec{r}_1(t) = \vec{r}_2(s)} \iff \langle 1+t, -2+3t, 4-t\rangle = \langle 2s, 3+s, -3+4s\rangle$$

$$\iff \begin{cases} 1+t = 2s \\ 3t-2 = s+3 \\ 4-t = 4s-3 \end{cases} \iff \begin{cases} t = 2s-1 \\ 3(2s-1)-2 = s+3 \\ 4-(2s-1) = 4s-3 \end{cases} \iff \begin{cases} t = 2s-1 \\ 6s-3-2 = s+3 \\ 4-2s+1 = 4s-3 \end{cases}$$

$$\iff \begin{cases} t = 11/5 \\ s = 8/5 \\ 6s = 8 \end{cases} \iff \begin{cases} t = 11/5 \\ s = 8/5 \\ s = 8/6 \end{cases} \neq$$

∴ There is no intersection point! ////

ⓑ $\vec{r}_1(t) = \langle -2,-1,3\rangle + t\langle 1,1,1\rangle$, $\vec{r}_2(t) = \langle 0,-1,1\rangle + t\langle 1,0,-1\rangle$

Sol: $\vec{r}_1(t) = \langle t-2, t-1, t+3\rangle$, $\vec{r}_2(t) = \langle t, -1, 1-t\rangle$

$$\vec{r}_1(s) = \vec{r}_2(t) \iff \langle s-2, s-1, s+3\rangle = \langle t, -1, 1-t\rangle$$

$$\iff \begin{cases} s-2 = t \\ s-1 = -1 \\ s+3 = 1-t \end{cases} \iff \begin{cases} t = -2 \\ s = 0 \\ 3 = 1-(-2) \end{cases} \iff \begin{cases} s = 0 \\ t = -2 \\ 3 = 3 \leftarrow \text{True!} \end{cases}$$

$$\iff \begin{cases} s = 0 \\ t = -2 \end{cases}$$

Hence $\vec{r}_1(t)$ intersects $\vec{r}_2(t)$ at $\vec{r}_1(0) = \langle -2,-1,3\rangle$. ////

Planes in $\mathbb{R}^3$ are given by a point (x_0, y_0, z_0) in the plane and a nonzero **normal vector** $\vec{n}$ to which all vectors the (homogeneous) plane are orthogonal.

The plane P through point (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c \rangle$ can be expressed as...

① $P = \{ \vec{x} : \vec{n} \cdot (\vec{x} - \vec{p}) = 0 \}$ ← Set (set notation)

② $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ ← equations (vector equation)

③ $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ (scalar equation)

Ex: Compute vector and scalar equations for the plane...

ⓐ through point $(-1, \frac{1}{2}, 3)$ with normal vector $\langle 1, 4, 1 \rangle$.

Sol: $\vec{n} = \langle 1, 4, 1 \rangle$, $\vec{p} = \langle -1, \frac{1}{2}, 3 \rangle$

Vector Equation: $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ i.e. $\langle 1, 4, 1 \rangle \cdot \langle x - (-1), y - \frac{1}{2}, z - 3 \rangle = 0$

scalar equation: $1(x - (-1)) + 4(y - \frac{1}{2}) + 1(z - 3) = 0$

$\iff x + 4y + z = 4$ ☐

ⓑ through the points $(2, 1, 2)$, $(3, -8, 6)$, $(-2, -3, 1)$.
(labelled P, Q, R)

Sol: $\vec{PQ} = \langle 3-2, -8-1, 6-2 \rangle = \langle 1, -9, 4 \rangle$, $\vec{QR} = \langle -2-2, -3-1, 1-2 \rangle = \langle -4, -4, -1 \rangle$

$$\vec{n} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \left\langle \begin{vmatrix} -9 & 4 \\ -4 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -9 \\ -4 & -4 \end{vmatrix} \right\rangle = \langle 25, -15, -40 \rangle$$

Vec EQ: $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ i.e. $\langle 25, -15, -40 \rangle \cdot \langle x + 2, y + 3, z - 1 \rangle = 0$

Scalar Eq: $25(x + 2) - 15(y + 3) - 40(z - 1) = 0$

or: $5(x + 2) - 3(y + 3) - 8(z - 1) = 0$ ☐

Ex: Compute plane through the point $(5,3,5)$ and parallel to the plane $x+y+z=3$.

Sol: $\vec{n} = \langle 1,1,1 \rangle$ ← parallel to $x+y+z=3$.

$\vec{p} = \langle 5,3,5 \rangle$

$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

∴ the desired plane has equation $\langle 1,1,1 \rangle \cdot \langle x-5, y-3, z-5 \rangle = 0$

i.e. $(x-5) + (y-3) + (z-5) = 0$

i.e. $x + y + z = 13$

$\square$

Ex: The plane passing through the point $(3,1,4)$ and containing the line of intersection of the planes

$x + 2y + 3z = 1$ and $2x - y + z = -3$.

Sol: First: Compute the line.

$$\begin{cases} x + 2y + 3z = 1 \\ 2x - y + z = -3 \end{cases} \iff \begin{cases} 2x + 4y + 6z = 2 \\ -5y - 5z = -5 \end{cases} \iff \begin{cases} x + 2y + 3z = 1 \\ y = 1 - z \end{cases}$$

$$\iff \begin{cases} x + 2(1-z) + 3z = 1 \\ y = 1 - z \end{cases} \iff \begin{cases} x + z = -1 \\ y = 1 - z \end{cases} \iff \begin{cases} x = -1 - z \\ y = 1 - z \end{cases}$$

∴ the line is $\vec{r}(t) = \langle -1, 1, 0 \rangle + t \langle -1, -1, 1 \rangle$. Can obtain two more points: $t=0$: $(-1,1,0)$, $t=-1$: $(0,2,-1)$

Two vectors for the plane: $\vec{u} = \langle 3-(-1), 1-1, 4-0 \rangle = \langle 4,0,4 \rangle$

$\vec{v} = \langle 3-0, 1-2, 4-(-1) \rangle = \langle 3,-1,5 \rangle$

$$\vec{n} = \left(\tfrac{1}{4}\vec{u}\right) \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 3 & -1 & 5 \end{vmatrix} = \left\langle \begin{vmatrix} 0 & 1 \\ -1 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \right\rangle = \langle 1, -2, -1 \rangle$$

∴ The plane has equation $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ i.e. $(x-3) - 2(y-1) - (z-4) = 0$

$\square$

Ex: Find the intersection point of the line

$5x = y/2 = z+2$ and the plane $10x - 7y + 3z + 24 = 0$.

Sol: the line has symmetric equations $\frac{x-0}{1/5} = \frac{y-0}{2} = \frac{z-(-2)}{1}$, so it has vector equation $\vec{r}(t) = \langle 0, 0, -2\rangle + t\langle \frac{1}{5}, 2, 1\rangle$

$= \langle \frac{1}{5}t, 2t, t-2\rangle$ ←

At the intersection we have

$5(\frac{1}{5}t) - 7(2t) + 3(t-2) + 24 = 0$

$\Leftrightarrow$ $t - 14t + 3t - 6 + 24 = 0$ $\Leftrightarrow$ $-10t + 18 = 0 \Leftrightarrow t = \frac{9}{5}$.

Hence the point of intersection is $(\frac{9}{25}, \frac{18}{5}, -\frac{1}{5})$, because $\vec{r}(9/5) = \langle \frac{9}{25}, \frac{18}{5}, -\frac{1}{5}\rangle$.

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Ex: Are the given planes parallel, orthogonal, or neither?

(a) $x + 4y - 3z = 1$ and $-3x + 6y + 7z = 0$

$\vec{n}_1 = \langle 1, 4, -3\rangle$ $\quad$ $\vec{n}_2 = \langle -3, 6, 7\rangle$

$\vec{n}_2 = c\vec{n}_1 \Rightarrow \langle -3, 6, 7\rangle = \langle c, 4c, -3c\rangle$

$\Rightarrow -3 = c$ and $c = \frac{3}{2}$ ✗ absurd!

∴ these planes are not parallel.

$\vec{n}_1 \cdot \vec{n}_2 = -3 + 24 - 21 = 0$ ∴ The planes are orthogonal!

(b) $9x - 3y + 6z = 2$ and $2y = 6x + 4z \rightsquigarrow 0 = 6x - 2y + 4z$

$\vec{n}_1 = \langle 9, -3, 6\rangle = 3\langle 3, -1, 2\rangle$ $\quad$ $\vec{n}_2 = \langle 6, -2, 4\rangle = 2\langle 3, -1, 2\rangle$

∴ $\vec{n}_1 = \frac{3}{2}\cdot 2\langle 3, -1, 2\rangle = \frac{3}{2}\vec{n}_2$.

∴ The planes are parallel.

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